# Indian Statistical Institute, Bangalore Centre.
# End-Semester Exam : Probability 1

Instructor : Yogeshwaran D.

Date : December 31st, 2021.

**Max. points : 40.** **Time Limit : 2.5 hours.**

**Submit solutions via Moodle by 12.30 PM on December 31st.**

**Please write your name and the honesty statement below on your answer script and sign below the same. Else 2 points will be deducted.**

*I have not received, I have not given, nor will I give or receive, any assistance to another student taking this exam, including discussing the exam with other students. The solution to the problems are my own and I have not copied it from anywhere else. I have used only class notes, notes from TA sessions, my own notes and assignment solutions.*

**There are two parts to the question paper - PART A and PART B. Read the instructions for each section carefully.**

## 1 PART A : MULTIPLE-CHOICE QUESTIONS - 10 Points.

**Please write only the correct choice(s) (for ex., (a), (b) et al.) in your answer scripts. No explanations are needed. Write PART A answers in a separate page.**

**Some questions will have multiple correct choices. Answer all questions. Each question carries 2 points.**

1. Let a standard die be thrown twice. Let $X$ denote the value of the first throw and $Y$ the value of second throw. Which of the following is true ?
   (a) $X$ and $Y$ are not equal in distribution i.e., their pmfs are different.
   (b) $X$ and $Y$ are equal in distribution i.e., their pmfs are the same.
   (c) $\mathbb{E}[X] = 7/2$ and $\mathbb{E}[Y] = 3$.
   (d) $\mathbb{E}[X] = 7/2$ and $\mathbb{E}[Y] = 7/2$.
   (e) $\mathbb{E}[Y] = 7/2$ and $\mathbb{E}[X] = 3$.
   (f) $\mathbb{E}[X] = 3$ and $\mathbb{E}[Y] = 3$.

2. Let $p, q$ be two pmfs on a finite sample space $S$ of cardinality at least two. Then which of the following statements are correct?

(a) $p^2$ is always a pmf.
(b) $p^2/2$ is always a pmf.
(c) $p^2 + q^2$ is always a pmf.
(d) $\frac{p+q}{2}$ is always a pmf.
(e) $\frac{p}{2}$ is never a pmf.
(f) None of the above.

3. Suppose $A$ and $B$ are two events such that $A \subset B$, and $\mathbb{P}\{A\}, \mathbb{P}\{B\} \in (0,1)$. Then which of the following statements are true?

(a) $\mathbb{P}\{B \mid A\} = 1$.
(b) $\mathbb{P}\{A \mid B^c\} = \mathbb{P}\{B^c \mid A\}$.
(c) $\mathbb{P}\{A \mid B\} = 1 - \mathbb{P}\{A^c \mid B\}$.
(d) $\mathbb{P}\{A^c \mid B^c\} = 1$.
(e) None of the above.

4. Which of the following is a distribution function ?

(a) $F(x) = e^{-x}$
(b) $F(x) = \frac{1}{1+x^2}$
(c) $F(x) = 1 - \frac{e^{-x}}{2}$
(d) $F(x) = 1 - 2e^{-|x|}$
(e) None of the above.

5. Let $X$ be a continuous random variable. Let $Y = X + 1, Z = X^2$ be two random variables. Which of the following hold ?

(a) $Y$ is a continuous random variable but $Z$ is not.
(b) $Y, Z$ are both continuous random variables.
(c) $f_Z(z) = 0$ for $z \leq 0$.
(d) $Z$ is a continuous random variable but $Y$ is not.
(e) $f_Y(y) = 0$ for $y \leq 1$.

## 2 PART B : 30 Points.

**Answer any three questions only. All questions carry 10 points.**

**Give necessary justifications and explanations for all your arguments. If you are citing results from the class, mention it clearly. Always define the underlying random variables and events clearly before computing anything !**

1. Fix $n \geq 1$ and suppose that $n$ is divisible by $k_1, \ldots, k_m$ where $k_1, \ldots, k_m$ are distinct natural numbers. We pick a random number $j$ uniformly from $[n]$. Let $A_k$ the event that $j$ is divisible by $k$.

(a) Compute $\mathbb{P}(A_{k_1} \cap A_{k_2} \cap \ldots \cap A_{k_n})$ for any $k_1, k_2, \ldots, k_m$ (not necessarily co-prime). **(3)**

(b) Let $k_1, \ldots, k_3$ be pairwise coprime (i.e., $gcd(k_1, k_2), gcd(k_2, k_3), gcd(k_1, k_3) = 1$). Prove that $A_{k_1}, \ldots, A_{k_3}$ are independent. (**4**)

(c) If $k_1, \ldots, k_3$ are coprime (i.e., $gcd(k_1, k_2, k_3) = 1$) but not pairwise comprime then are $A_{k_1}, A_{k_2}, A_{k_3}$ independent ? (**3**)

2. Consider the Polya urn model where initially there are $r$ red balls and $b$ black balls respectively where $r, b \in \mathbb{N}$. Let $c \in \mathbb{N}$. At each step, a ball is chosen uniformly at random and $c$ balls of the same colour are added into the urn along with the chosen ball. No ball of the opposite colour are added. Let $X_{l,m,n}$ denote the indicator random variable that a red ball is chosen at the $l$th, $m$th and $n$th step where $l < m < n$. Compute $\mathbb{E}[X_{l,m,n}]$, $\mathsf{VAR}(X_{l,m,n})$ for all $l, m, n \geq 1$.

3. An engineering system consisting of $n$ components is said to be a $k$-out-of-$n$ system ($k \leq n$) if the system functions if and only if at least $k$ out of the $n$ components function. Suppose that all the components function independently.

   (a) Consider a 3-out-of-5 system and say that the $i$th component functions with probability $p_i, i = 1, 2, 3, 4$. Compute the probability that the system functions. (**3**)

   (b) Consider a $k$-out-of-$n$ system where all the components function independently with probability $p$. Compute the probability that the system functions. (**2**)

   (c) Consider a $k$-out-of-$n$ system with $k \geq 2$ and where all the components function independently with probability $p$. Find the conditional probability that either the first or last components (i.e., one of two components function) are functioning given that the system functions. (**5**)

4. Let $b \in (0, \infty)$ be a parameter. Suppose that $X$ is a random variable with pdf $f(x) := \frac{1}{2b} e^{-\frac{|x|}{b}}$ for $x \in \mathbb{R}$.

   (a) Find the pdf and CDF of the random variable $|X|$. (**5**)

   (b) Compute all the moments of $X$ i.e., $\mathbb{E}\left[X^k\right]$ for all $k \geq 1$. (**5**)

5. Let $\alpha > 0$ and $X$ be a random variable with the pdf given by

$$f(x) = \frac{\alpha}{x^{\alpha+1}}, 1 \leq x < \infty \; ; \; f(x) = 0, \, x < 1.$$

   (a) Find pdf and CDF of the following random variable $X_1 = X^2$. (**2**)

   (b) Find pdf and CDF of the following random variable $X_1 = X^{-1}$. (**3**)

   (c) Find $k \in \mathbb{N}$ such that $\mathbb{E}\left[X_1^k\right]$ is well-defined and compute $\mathbb{E}\left[X_1^k\right]$ in such cases. (**5**)